Continuity 2

Given the following function:

$$\frac{x^2 - xy}{x + y}$$

1. Compute the iterated and radial limits. Can it be concluded that the double limit exists?
2. Now consider the following piecewise function:

$$f(x, y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ a - 10 & \text{if } (x, y) = (0, 0) \end{cases}$$

   Find the value of a such that the function is continuous at $(0, 0)$.

1. Iterated limits:

$$L_1 = \lim_{y\to 0}\left[\lim_{x\to 0}\frac{x^3-xy^2}{x^2+y^2}\right] = 0$$

$$L_2 = \lim_{x\to 0}\left[\lim_{y\to 0}\frac{x^3-xy^2}{x^2+y^2}\right] = \lim_{x\to 0}\frac{x^3}{x^2} = \lim_{x\to 0} x = 0$$

   Radial limit:

$$y = mx$$

$$\lim_{x\to 0}\frac{x^3-x^3m^2}{x^2+m^2x^2} = \lim_{x\to 0}\frac{x(1-m^2)}{1+m^2} = 0$$

2. For the function to be continuous, the function evaluated at the point must be equal to the limit. We calculate the limit:

$$\lim_{(x,y)\to(0,0)}\frac{x^3-xy^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x\frac{x^2-y^2}{x^2+y^2}$$

   We see that $\frac{x^2-y^2}{x^2+y^2}$ is bounded:

$$x^2+y^2 \geq x^2-y^2$$

$$1 \geq \frac{x^2-y^2}{x^2+y^2}$$

   Furthermore:

$$-x^2-y^2 \leq x^2-y^2$$

$$-(x^2+y^2) \leq x^2-y^2$$

$$-1 \leq \frac{x^2-y^2}{x^2+y^2}$$

   Therefore:

$$-1 \leq \frac{x^2-y^2}{x^2+y^2} \leq 1$$

   And by the theorem of the limit of an infinitesimal times a bounded function:

$$\lim_{(x,y)\to(0,0)} x\frac{x^2-y^2}{x^2+y^2} = 0$$

   Therefore, for the function to be continuous, it must hold that $a = 10$.